1. Measurements and error calculus

1.1 Introduction

The goal of this laboratory course is to introduce the notions of carrying out an experiment, acquiring and writing up the data, and finally analyzing the results. By means of a simple mathematical pendulum the gravitational acceleration g shall be determined. Various measurement methods and the corresponding error sources will be discussed.
Some keywords related to the present experiment are:

- precision and accuracy of reading of measurement devices,
- book keeping and analysis of measured data,
- systematic and statistical errors,
- histograms of data,
- calculation of mean values and corresponding errors.

1.2 Theoretical background

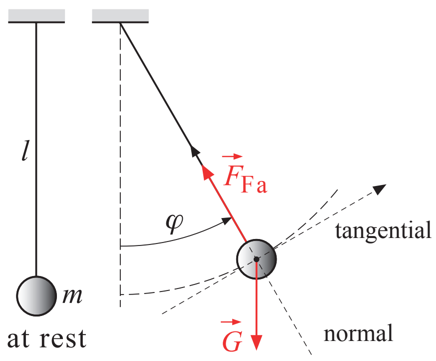

Figure 1.1: The pendulum.

A mass m be suspended by means of a thread, the length l of which be much greater than the diameter of the mass. We assume that friction may be neglected. For small oscillation amplitudes the period of time T is independent of mass and amplitude; the period solely depends on the length of the pendulum and on the gravitational acceleration g (see Appendix):

$$T = 2\pi\sqrt{\frac{l}{g}} \qquad \text{or} \qquad g = \frac{4\pi^2 l}{T^2} \tag{1.1}$$

Thus, the measurement of g can be reduced to simple measurements of the length and of the period of oscillation of a pendulum.

1.3 Experiment

1.3.1 Goal of the experiment

- determination of the gravitational acceleration g from the length and the period of oscillation of a pendulum,
- comparison of several ways to measure time,
- discussion of error propagation and finally the calculation of the error on g.

1.3.2 Measurements

- Measure the length of the pendulum using the ruler and estimate the accuracy of the measurement.
- Time measurement (c.f. appendix on error calculus): at constant length of the pendulum and for small amplitudes ($\varphi = 5° - 10°$ corresponding to $\varphi = 0.09 - 0.18$ radians) measure the period of oscillation in the following ways:

  (a) Measure several times the temporal period T over a single cycle of oscillation using a standard stopwatch. Calculate the mean value $\overline{T}$.

  (b) The same as in (a) but using an electronic stopwatch.

  (c) Using the standard mechanical stopwatch, measure several times the time duration of five cycles of oscillation. Calculate T and the mean value $\overline{T}$.

  (d) The same as in (c) but using the electronic stopwatch. Calculate again T and $\overline{T}$.

- Enter the results into the tables in Section 1.3.5.
- Calculate the statistical errors on the mean values m_T and the relative error r_T.
- Discuss the different results under the following aspects:
  - In which case does the calculation of the statistical errors make sense ? In which case does an estimation of the measurement error yield more reasonable results ? Give the reasons of your claims.
  - Usually, methods (c) or (d) are chosen for measuring the time of oscillation. Why should (c) and (d) be preferred over (a) and (b) ?

1.3.3 Analysis of the measurements

- Calculate g from the mean value of T and the length l using Eq. 1.1:

$$g = \frac{4\,\pi^2\,l}{\overline{T}^2} \tag{1.2}$$

- Calculate the error on g from the *relative* errors on T and l using the corresponding laws of error propagation (c.f. Appendix on Error Calculus).
- Verify the relation $T \propto \sqrt{l}$: Measure the time of oscillation of the pendulum for different lengths. Plot T as a function of $\sqrt{l}$. Which kind of function do you expect ?
- Determine the gravitational acceleration g from the slope of the graph.

1.3.4 If there is enough time left...

- Dependence of the period on the amplitude of the oscillation:
  - You are about to measure a small effect: which method is best suited for this kind of precise measurements ?
  - Measure the time of oscillation for three different initial amplitudes and enter the values into the corresponding table.

1.3.5 Data and results

Determination of g

length: $l =$ uncertainty: $m_l =$

period of time for one single cycle:

	stopwatch T (s)	elec. watch T (s)
1.		
2.		
3.		
4.		
5.		
6.		
$\overline{T} =$		
$m_{\overline{T}}$		
$r_{\overline{T}} = \frac{m_{\overline{T}}}{\overline{T}}$		

measurement over five cycles of oscillation:

	mech. watch $5T$ (s)	mech. watch T (s)	elec. watch $5T$ (s)	elec. watch T (s)
1.				
2.				
3.				
$\overline{T} =$	XXX		XXX	
$m_{\overline{T}}$	XXX		XXX	
$r_{\overline{T}}$	XXX		XXX	

Dependence of T on the length of the pendulum

	length l (m)	$5T$(s)	T(s)
1.			
.			
.			
2.			
.			
.			
3.			
.			
.			
4.			
.			
.			

Dependence of T on the initial amplitude

	amplitude	$5T$ (s)	T (s)
1.			
.			
.			
2.			
.			
.			
3.			
.			
.			

1.4 Appendix

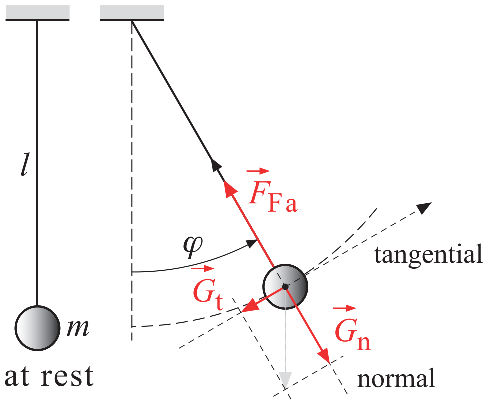

Figure 1.2: The pendulum.

The mass m issuspended by means of a thread, the length l of which be much greater than the diameter of the mass. We consider the mass to be a point mass. Moreover, we will neglect friction. The angle $\varphi(t)$ is chosen as coordinate describing the state of the pendulum. We separate the horizontal and vertical components of the forces, and the equation of motion reads:

$$\text{equation of motion: } m\vec{a} = \vec{F}_{Fa} + \vec{G}$$

$$\text{tangential}: \quad ml\,\tfrac{d^2\varphi}{dt^2} = -mg\,\sin\varphi \tag{1.3}$$

$$\text{normal}: \quad ml\,[\tfrac{d\varphi}{dt}]^2 = F_{\text{Fa}} - mg\,\cos\varphi \tag{1.4}$$

Both equations contain the force along the thread $F_{Fa}(t)$, which a priori is unknown, as well as the coordinate $\varphi(t)$, which is to be determined. The latter can be deduced from Eq. 1.3: since this differential equation is transcendental and cannot be solved for φ, we replace the sin-term in Eq. 1.3 by φ, the approximation being valid for small angles ($\varphi \ll \pi/2$): $\sin\varphi \approx \varphi + ...$ and therefore

$$ml\,\frac{d^2\varphi}{dt^2} = -mg\,\varphi \qquad \text{or} \qquad \frac{d^2\varphi}{dt^2} + \frac{g}{l}\,\varphi = 0 \tag{1.5}$$

According to experience, the motion of the pendulum is periodic. Therefore, the following *ansatz* is chosen for describing this periodic motion:

$$\varphi(t) = \varphi_0\,\cos(\omega t + \delta) \qquad \begin{aligned} \varphi_0 &= \text{amplitude} \\ \omega &= 2\pi\nu = 2\pi/T = \text{angular frequency} \\ \delta &= \text{constant phase offset} \end{aligned} \tag{1.6}$$

φ_0 and δ depend on the initial state of the pendulum, and ω is determined by inserting the *ansatz* Eq. 1.6 into the differential equation Eq. 1.5. Differentiating twice Eq. 1.6 yields:

$$\frac{d^2\varphi}{dt^2} = -\varphi_0\,\omega^2\cos(\omega t + \delta) \tag{1.7}$$

We insert this into Eq. 1.5 and get:

$$-\varphi_0\,\omega^2\cos(\omega t + \delta) + \frac{g}{l}\,\varphi_0\,\cos(\omega t + \delta) = 0 \qquad \Rightarrow \qquad \omega = \sqrt{\frac{g}{l}} = \frac{2\pi}{T}. \tag{1.8}$$

The period of oscillation is – for small amplitudes and point masses – independent of mass and amplitude. For larger amplitudes, the motion still is periodic in time but becomes anharmonic; the period of oscillation increases with increasing amplitude due to higher orders appearing in Eq. 1.5.